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LORENTZ GAUGE FIXING AND LATTICE QED ¹

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Abstract. The Gribov ambiguity problem is studied for compact lattice QED within the Lorentz gauge. In the Coulomb phase, Gribov copies are mainly caused by double Dirac sheets and zero-momentum modes of the gauge fields. Removing them by (non-) periodic gauge transformations allows to reach the absolute extremum of the Lorentz gauge functional. For standard Lorentz gauge fixing the Wilson fermion correlator turns out to be strongly effected by the zero-momentum modes. A reliable fermion mass estimate requires the proper treatment of these modes.

1. Introduction

Most of the applications of lattice gauge theories are based on and are employing their manifest gauge invariance. However, in order to get a better understanding of the structure of the lattice theory itself and to interpret correctly results obtained in Monte Carlo simulations, it is instructive to compare also gauge variant quantities such as gauge and fermion field propagators with corresponding analytic perturbative results. In this respect, compact lattice QED within the Coulomb phase serves as a very useful ‘test ground’. In the weak coupling limit this theory is supposed to describe non-interacting massless photons.

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In order to fix the gauge the Lorentz (or Landau) gauge condition is normally applied. For non-Abelian gauge theories there is no unique solution, i.e. so-called Gribov copies occur [1]. Within continuum QED such a problem arises, too, if the theory is defined on a torus [2]. The lattice discretization may cause additional problems. Indeed, various lattice studies [3 – 11] have revealed nontrivial effects. The standard Lorentz (or Landau) gauge fixing procedure leads to a τ -dependence of the non-zero-momentum transverse photon correlator inconsistent with the expected zero-mass behavior [3]. Numerical [4, 5, 9] and analytical [7] studies have shown that there is a connection between ‘bad’ gauge (or Gribov) copies and the appearance of periodically closed double Dirac sheets (DDS). The removal of DDS by appropriate gauge transformations restores the correct perturbative behavior of the photon correlator at non-zero momentum, but it does not completely resolve the Gribov ambiguity problem. Gribov copies related to other local extrema of the gauge functional and connected with zero-momentum modes (ZMM) of the gauge fields still appear. They ‘damage’ gauge dependent observables such as the zero-momentum gauge field correlator [7, 9, 8] and the fermion propagator [10, 11], respectively.

There is a special Lorentz gauge, for which both the double Dirac sheets and the zero-momentum modes can be removed from the gauge fields. We call it *zero-momentum Lorentz gauge* (ZML) [8]. It allows to reach the global extremum of the Lorentz gauge functional in almost 100% of the cases. In comparison with the standard Lorentz gauge procedure (LG) it demonstrates very clearly the strong effects caused by the zero-momentum modes.

In the given talk we are going to review the results of [8, 11] with special emphasis on the question, how Gribov copies influence the Wilson-fermion propagator within the Coulomb phase of quenched QED. We want to show that a reliable estimate of the (renormalized) fermion mass requires either the removal of the zero-momentum modes or their proper perturbative treatment, when comparing the numerical results with analytic expressions. By employing the ZML-gauge we shall estimate the fermion mass in agreement with standard perturbation theory.

2. The Action and the Observables

We consider 4d compact QED in the quenched approximation on a finite lattice ($V = N_s^3 \times N_t$). The pure gauge part of the standard Wilson action [12] reads

$$S_G = \beta \sum_{x, \mu < \nu} (1 - \cos \theta_{x, \mu\nu}), \quad (1)$$

with the plaquette angle $\theta_{x,\mu\nu} = \theta_{x,\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x,\nu}$ related to the link variables $\theta_{x,\mu} \in (-\pi, \pi]$. $\beta = 1/e_0^2$ is the inverse bare coupling. The lattice spacing is put $a = 1$, for simplicity.

The fermion part is given by

$$S_F = \sum_{x,y} \bar{\psi}_x \mathbf{M}_{xy}(\theta) \psi_y, \quad \mathbf{M} = \mathbf{1} - \kappa \mathbf{D}, \quad (2)$$

$$\mathbf{D}_{xy} = \sum_{\mu=1}^4 \left\{ U_{x,\mu} P_{\mu}^{-} \delta_{y,x+\hat{\mu}} + U_{x-\hat{\mu},\mu}^* P_{\mu}^{+} \delta_{y,x-\hat{\mu}} \right\},$$

where $P_{\mu}^{\pm} = \hat{1} \pm \gamma_{\mu}$ and $U_{x,\mu} = e^{i\theta_{x,\mu}}$. The hopping-parameter κ is related to the bare mass m_0 by $\kappa = 1/(8 + 2m_0)$.

In quenched QED the observables have to be averaged with respect to the gauge field $\{\theta\}$ with the weight $\exp(-S_G)$. We imply periodic boundary conditions (b.c.) except for the fermion fields, which we choose to be anti-periodic in the $x_4 \equiv \tau$ -direction.

The first gauge variant observable we are going to discuss is the transverse photon correlator at non-zero momentum

$$\Gamma_T^{\text{ph}}(\vec{p}; \tau) = \langle \Phi(\vec{p}; \tau) \Phi^*(\vec{p}; 0) \rangle, \quad (3)$$

$$\Phi(\vec{p}; \tau) = \sum_{\vec{x}} \exp(i\vec{p}\vec{x} + \frac{i}{2}p_{\mu}) \sin \theta_{\vec{x}\tau,\mu}$$

with $(\mu = 1, 3, \vec{p} = (0, p, 0))$. The second one is the fermion propagator. For a given gauge field $\{\theta\}$ we have

$$\Gamma(\tau) = \frac{1}{V} \sum_{\vec{x}, x_4} \sum_{\vec{y}} \mathbf{M}_{\vec{x}, x_4; \vec{y}, x_4 + \tau}^{-1}(\theta). \quad (4)$$

In the following we shall restrict ourselves to the vectorial part

$$\Gamma_V(\tau) = \frac{1}{4} \text{Re Tr} (\gamma_4 \Gamma(\tau)), \quad (5)$$

with the trace taken with respect to the spinor indices. For the b.c. mentioned above, $\langle \Gamma_V(\tau) \rangle$ is an even function of $\tau - N_t/2$.

Later on, we shall compare the expectation value $\langle \Gamma_V \rangle$ with the result of a simple approximation, which takes only constant gauge field modes into account. The correlator in a uniform background $\theta_{x,\mu} \equiv \phi_{\mu}$, $-\pi < \phi_{\mu} \leq \pi$, $\mu = 1, \dots, 4$ can be represented as

$$\Gamma_V(\tau; \phi) = \frac{1 - \delta_{\tau,0}}{2(1 + \mathcal{M})} \times \frac{[\mathcal{E}^{\tau} + \mathcal{E}^{2N_t - \tau}] \cos(\phi_4 \tau) + [\mathcal{E}^{N_t + \tau} + \mathcal{E}^{N_t - \tau}] \cos[\phi_4(N_t - \tau)]}{1 + \mathcal{E}^{2N_t} + 2\mathcal{E}^{N_t} \cos(\phi_4 N_t)}, \quad (6)$$

where

$$\mathcal{E} = 1 + \frac{\mathcal{M}^2 + \mathcal{K}^2}{2(1 + \mathcal{M})} - \frac{\sqrt{\mathcal{M}^2 + \mathcal{K}^2} \sqrt{(\mathcal{M} + 2)^2 + \mathcal{K}^2}}{2(1 + \mathcal{M})} ;$$

$$\mathcal{M} = m_0 + \sum_{l=1}^3 (1 - \cos \phi_l), \quad \mathcal{K} = \sqrt{\sum_{l=1}^3 \sin^2 \phi_l}, \quad m_0 > 0.$$

For $\phi_\mu = 0$, $\mu = 1, \dots, 4$ the free fermion correlator for finite lattice size [13] is reproduced.

3. Lorentz Gauge Fixing

In numerical simulations the Lorentz gauge is fixed by iteratively maximizing the gauge functional

$$F(\theta) = \frac{1}{V_4} \sum_x F_x(\theta) ; \quad F_x(\theta) = \frac{1}{8} \sum_{\mu=1}^4 [\cos \theta_{x\mu} + \cos \theta_{x-\hat{\mu};\mu}] \quad (7)$$

with respect to the (local) gauge transformations

$$U_{x\mu} \longrightarrow \Lambda_x U_{x\mu} \Lambda_{x+\hat{\mu}}^* ; \quad \Lambda_x = \exp\{i\Omega_x\} \in U(1) . \quad (8)$$

The algorithm is called standard Lorentz gauge fixing (LG), if it consists only of local maximization and overrelaxation steps [14] with respect to gauge transformations periodic in space-time. The standard procedure gets normally stuck into *local* maxima of the gauge functional (7) (gauge copies).

It has been argued that the Gribov problem has to be solved by searching for the *global maximum* providing the best gauge copy [15]. In [8] we have shown that in order to reach the global maximum we have necessarily to suppress both the double Dirac sheets (DDS) and the zero-momentum modes (ZMM) in the gauge fields. Let us explain this in more detail.

DDS can be identified as follows. The plaquette angle $\theta_{x,\mu\nu}$ is decomposed into the gauge invariant (electro-) magnetic flux $\bar{\theta}_{x,\mu\nu} \in (-\pi, \pi]$ and the discrete gauge-dependent contribution $2\pi n_{x,\mu\nu}$, $n_{x,\mu\nu} = 0, \pm 1, \pm 2$ [16]. The latter represents a Dirac string passing through the given plaquette if $n_{x,\mu\nu} = \pm 1$ (*Dirac plaquette*). A set of Dirac plaquettes providing a world sheet of a Dirac string on the dual lattice is called *Dirac sheet*. DDS consist of two sheets with opposite flux orientation extending over the whole lattice and closing themselves by the periodic b.c. They can easily be identified by counting the total number of Dirac plaquettes $N_{DP}^{(\mu\nu)}$ for each choice $(\mu; \nu)$. The necessary condition for the occurrence of DDS is that at least for one of the six possibilities $(\mu; \nu)$ holds

$$N_{DP}^{(\mu\nu)} \geq 2 \frac{V}{N_\mu N_\nu} . \quad (9)$$

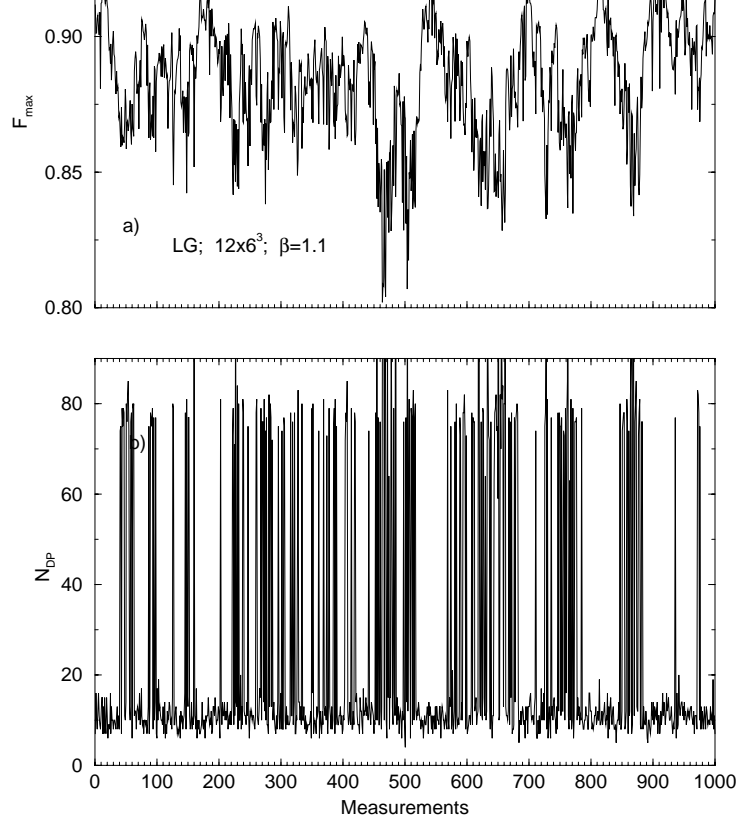


Figure 1. Time history of F_{max} (a) and N_{DP} (b) at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in the standard Lorentz gauge.

DDS can be removed by periodic gauge transformations.

The ZMM of the gauge field

$$\phi_\mu = \frac{1}{V} \sum_x \theta_{x,\mu} \quad (10)$$

do not contribute to the pure gauge field action either. For gauge configurations representing small fluctuations around constant modes it is easy to see, that the global maximum of the functional (7) requires $\phi_\mu \equiv 0$. The latter condition can be achieved by non-periodic gauge transformations

$$\theta_{x,\mu} \rightarrow \theta_{x,\mu}^c = c_\mu + \theta_{x,\mu} \mod 2\pi, \quad c_\mu \in (-\pi, \pi]. \quad (11)$$

We realize a proper gauge fixing procedure as proposed in [8]. Successive Lorentz gauge iteration steps are always followed by non-periodic gauge transformations suppressing the ZMM. Additionally we check, whether the gauge fields contain yet DDS. The latter can be excluded by repeating the procedure with initial random gauges. We call the combined procedure *zero-momentum Lorentz gauge* (ZML gauge). It yields the global maximum of the gauge functional with very high accuracy.

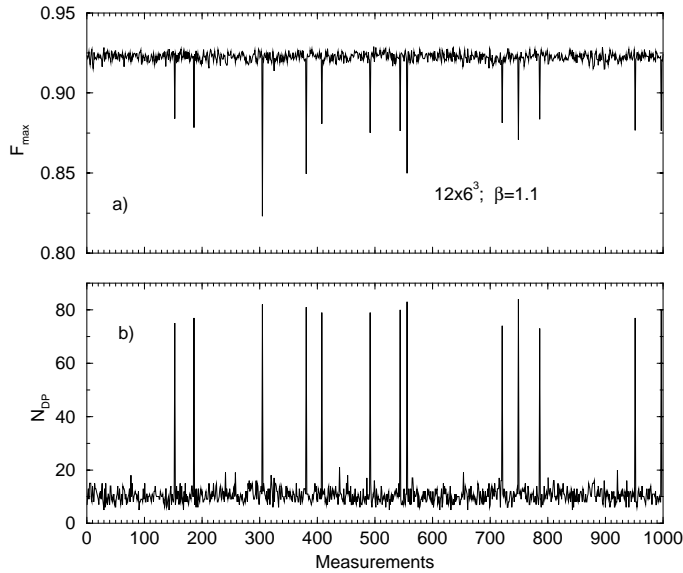


Figure 2. Time history of F_{max} (a) and N_{DP} (b) at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in ZML gauge.

In Figures 1 and 2 we show, how the achieved values of the gauge functional (7) are correlated with the occurrence of DDS visible as sharp peaks in the number of Dirac plaquettes. Whereas for LG strong fluctuations occur, they disappear after ZML gauge. The few DDS seen in Fig. 2 are easily removed by restarting the procedure with random initial gauges. Random gauges can also be used in order to convince oneself that the ZML gauge prescription leads to the global maximum of the gauge functional in more than 99% of the cases.

4. Results

First let us convince ourselves that the removal of the above mentioned gauge copies leads to the correct behaviour of the transverse photon propagator. In Fig. 3 we show the normalized correlator $\Gamma_T^{ph}(\vec{p}; \tau) / \Gamma_T^{ph}(\vec{p}; 0)$ for lowest non-vanishing momentum and for different Lorentz gauge prescriptions. For the standard one (LG) we see a clear deviation from the expected perturbative zero-mass result. We show also the result obtained with an axial Lorentz gauge (ALG) using an initial maximal-tree axial gauge con-

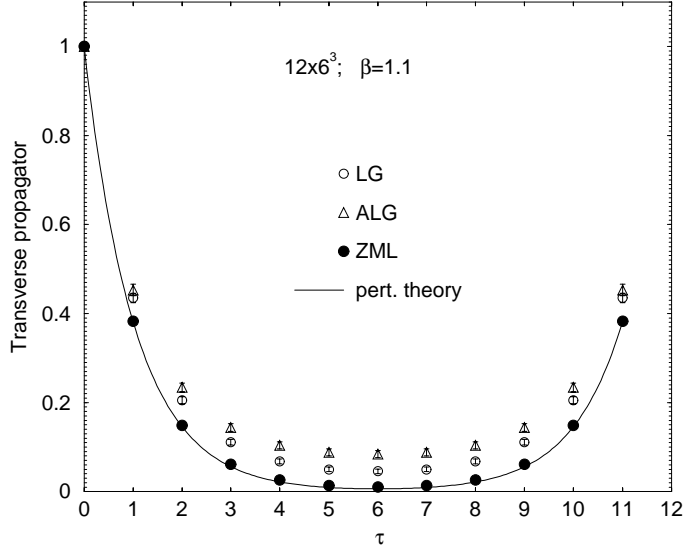


Figure 3. Transverse propagator at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in three different gauges.

dition [17], which provides a 'unique' prescription. The latter gauge fixing prescription turns out to be even worse! On the other hand the ZML gauge provides an excellent agreement with the perturbative result. In fact, as we convinced ourselves earlier, it is already sufficient to remove the DDS gauge copies in order to reach this agreement [4]. The given observations do not change, when β and/or the lattice size are increased considerably [9].

In the following we want to concentrate on the pure effect of the ZMM. Therefore, we compare the ZML gauge with a version of the standard Lorentz gauge, where the DDS are removed and the ZMM are left. We shall abbreviate the latter version also by LG. For both the LG and ZML gauges we have computed the averaged fermion correlator employing the conjugate gradient method and point-like sources. In the upper part of Fig. 4 we have plotted $\Gamma_V(\tau)$ (normalized to unity at $\tau = 1$). The situation seen is typical for a wide range of parameter values within the Coulomb phase. Obviously, there is a strong dependence of the fermion propagator on the gauge fixing procedure resulting in the presence or absence of ZMM. The masses to be extracted seem to have different values. Let us determine the effective mass $m_{eff}(\tau)$ in accordance with

$$\frac{\langle \Gamma(\tau + 1; \theta) \rangle_\theta}{\langle \Gamma(\tau; \theta) \rangle_\theta} = \frac{\cosh[E(\tau)(N_t/2 - \tau - 1)]}{\cosh[E(\tau)(N_t/2 - \tau)]} \quad (12)$$

where $E(\tau) = \ln(m_{eff}(\tau) + 1)$. See the lower part of Fig. 4. In the LG

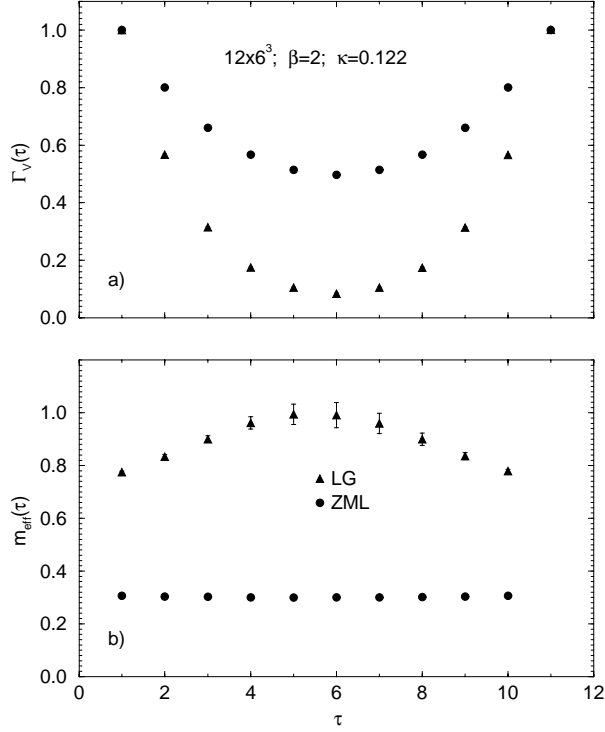


Figure 4. The fermion vector propagator (a) and the effective mass (b) at $\beta = 2$ and $\kappa = .122$ on a 12×6^3 lattice for LG and ZML gauges as explained in the text.

case no plateau is visible, whereas the ZML case provides a very stable one. Thus, only the ZML gauge yields a reliable mass estimate, whereas the standard method to fix the Lorentz gauge obviously fails.

To get deeper insight into the effect of ZMM for the LG case (with DDS suppressed) we measure the probability distributions $P(\phi)$ for the space- and time-like components of ZMM according to Eq. (10). The distributions turn out to be flat up to an effective cutoff at $|\phi_\mu| \simeq \pi/N_\mu$ and to be widely independent of β . In accordance with Eq. (6) we compute the fermion propagator for constant modes in the LG case and average

$$\langle \Gamma_V(\tau; \phi) \rangle_\phi = \int [d\phi] P(\phi) \Gamma_V(\tau; \phi). \quad (13)$$

The results for several parameter sets are presented in Fig. 5 together with the corresponding free (i.e. zero-background) propagator. We see clearly that the constant mode contributions strongly change the behavior of the fermion propagator and, naively speaking, produce a larger mass.

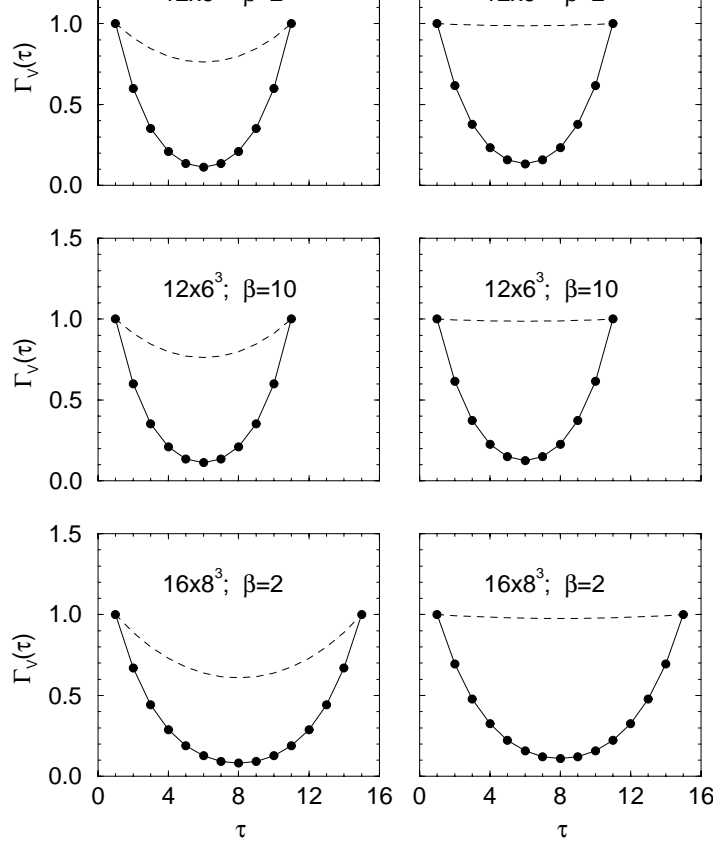


Figure 5. Free fermion propagator (dashed line) and averaged constant-mode propagator in the LG case (full line) for $\beta = 2, 10$, $\kappa = .120, .124$, lattice size 12×6^3 , 16×8^3 .

Finally, in Fig. 6 we present the fermion mass extracted from the vector fermion propagator within the ZML gauge for $\beta = 2.0$ and various κ -values. We see a nice linear behaviour from which by extrapolating to zero mass (solid line) we estimate the critical value $\kappa_c = 0.1307 \pm 0.0001$.

5. Conclusions

We have studied the effect of different gauge copies of the gauge field on gauge dependent correlators, in particular on the Wilson fermion propagator. We have convinced ourselves that the standard Lorentz gauge fixing prescription to maximize the functional (7) provides gauge copies with DDS and ZMM. These modes disturb the photon and the fermion correlator in comparison with perturbation theory and consequently spoil the (effective) mass estimate. A Lorentz gauge employing non-periodic gauge transformations in order to suppress the ZMM – additionally to DDS – (the ZML gauge) allows to reach the global maximum of the Lorentz gauge functional. Furthermore, it provides a reliable fermion mass determination, at least, if

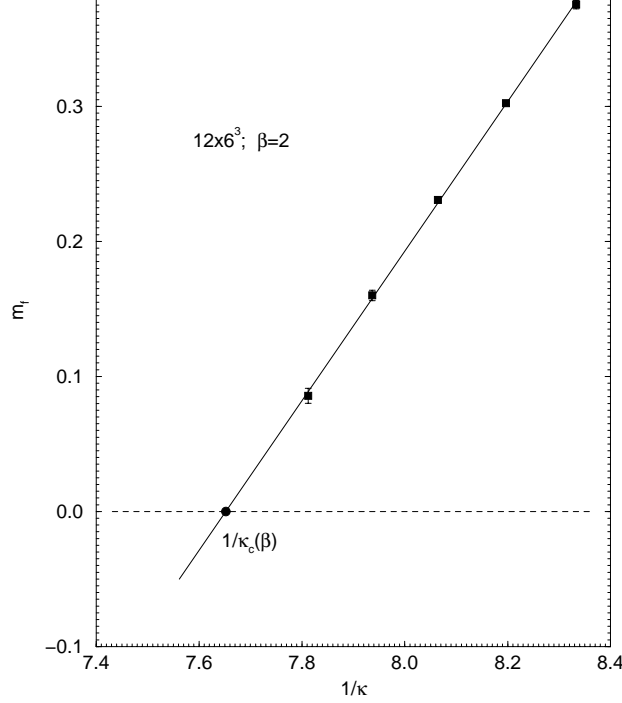


Figure 6. Fermion mass as a function of inverse κ obtained within the ZML gauge for $\beta = 2.0$ on a 12×6^3 lattice. The solid line represents a linear fit providing $\kappa_c(\beta) = 0.1307 \pm 0.0001$.

κ is chosen not too close to the chiral critical line $\kappa_c(\beta)$. A computation of the fermion propagator with constant background gauge fields taken from the ZMM of the quantum fields demonstrates the disturbing effect of these modes very clearly. Moreover, it shows the effect to be independent of the bare coupling and not to disappear for large volumes.

So far, we have studied the quenched approximation of U(1) lattice gauge theory. The gauge action (1) is invariant under non-periodic gauge transformations (11). Thus, we are allowed to use the ZML gauge for evaluating gauge dependent objects. Contrary to the gauge action, the fermionic part (2) does depend on the ZMM because of the (anti-) periodic boundary conditions. In this case another way of dealing with the Gribov problem has to be searched for.

The problems we have discussed here for compact QED show that gauge fixing has to be carried out and to be interpreted with care. This lesson has to be taken into account also in lattice QCD when extracting masses from gauge variant gauge and fermion correlators, respectively.

Acknowledgements

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